Hyperbolic Functions Cheat Sheet

The hyperbolic functions are a family of functions that are very similar to the trigonometric functions $\sin, \cos, \tan$ that you have been using throughout the A-level course. As a result, many of the identities and equations we will cover will look similar to their trigonometric counterparts. Hyperbolic functions are used to model many real-life scenarios; a common example can be seen when we consider a rope suspended between two points: if you let the rope hang under gravity, the shape that the rope naturally forms is known as a catenary, which is identical to the hyperbolic cosine function. In this chapter, we will familiarise ourselves with the hyperbolic functions and learn to use them in the same way as the trigonometric functions.

Definitions

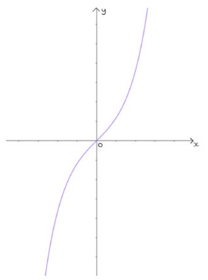
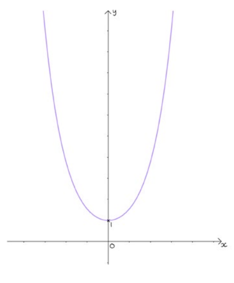
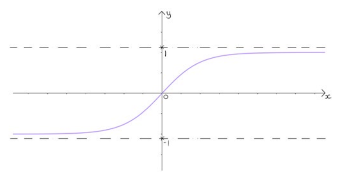
In Chapter 1. You learnt that $\sin, \cos$ and $\tan$ can be expressed in terms of e and i. The hyperbolic functions, however, are expressed only in terms of e.

- Hyperbolic sine, known as **sinh,** is defined as $\sinh(x) = \frac{e^x - e^{-x}}{2}$ (pronounced "shine" or "sinch")
- Hyperbolic cosine, known as **cosh**, is defined as $\cosh(x) = \frac{e^x + e^{-x}}{2}$ (pronounced "cosh")
- Hyperbolic tan, known as **tanh**, is defined as $\tanh(x) = \frac{\sinh x}{\cosh x} = \frac{e^{2x} - 1}{e^{2x} + 1}$ (pronounced "than" or "tanch")

These definitions tend to be useful when proving identities and solving equations involving hyperbolic functions.

Graphs

You also need to be able to sketch the graphs of the above hyperbolic functions.

Function	$y = \sinh(x)$	$y = \cosh(x)$	$y = \tanh(x)$
Graph			
Notes	For any a, $\sinh(-a) = -\sinh(a)$	For any a, $\cosh(-a) = \cosh(a)$	Asymptotes are $y = 1, y = -1$

Inverse hyperbolic functions

You also need to be able to use the inverse hyperbolic functions. Recall from Chapter 2 in Pure Year 2 that the inverse of a function is simply its reflection in the line $y = x$, and only exists if the function is one-to-one. The functions *sinh* and *tanh* are both one-to-one but *cosh* is not, so we must restrict its domain to $x \geq 0$ before we can look at its inverse. Here are the inverse functions you need to be familiar with, along with their domains:

- The inverse hyperbolic sine function is defined as $y = \operatorname{arsinh}(x),\ x \in \mathbb{R}$
- The inverse hyperbolic cosine function is defined as $y = \operatorname{arcosh}(x),\ x \geq 1$
- The inverse hyperbolic tangent function is defined as $y = \operatorname{artanh}(x),\ |x| < 1$

You can also express the inverse functions in terms of natural logarithms. These equivalences are very important, and you are expected to be able to prove them.

- $\operatorname{arsinh}(x) = \ln\left[x + \sqrt{x^2 + 1}\right]$
- $\operatorname{arcosh}(x) = \ln\left[x + \sqrt{x^2 - 1}\right],\ x \geq 1$
- $\operatorname{artanh}(x) = \frac{1}{2}\ln\left[\frac{1+x}{1-x}\right],\ |x| < 1$

These will be given to you in the formula booklet.

It is important to note that when solving equations of the form $\cosh x = k$ where $k > 1$, you will have two solutions: $x = \operatorname{arcosh}(k) = \ln\left[k \pm \sqrt{k^2 - 1}\right]$. The inverse function we defined above does not include both possibilities because we only considered $\cosh x$ for $x \geq 0$.

We will now prove one of the above statements. The proofs for the other two will use the same method.

Example 1: Prove that $\operatorname{arsinh}(x) = \ln\left[x + \sqrt{x^2 + 1}\right]$.

Let $y = \operatorname{arsinh} x$	$y = \operatorname{arsinh} x$
Take sinh of both sides	$x = \sinh y$
Use the exponential definition of sinh:	$x = \frac{e^y - e^{-y}}{2}$
Multiply by $2e^y$. This gives us a quadratic in e^y.	$2xe^y = e^{2y} - 1$ $e^{2y} - 2xe^y - 1 = 0$
Use the quadratic formula with $a = 1, b = -2x, c = -1$	$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$ $e^y = x \pm \sqrt{(x^2 + 1)}$
$x - \sqrt{(x^2+1)}$ is always negative since $\sqrt{(x^2+1)} > x$, so we can reject this solution as $e^y > 0$.	Reject $e^y = x - \sqrt{(x^2+1)}$ since $e^y > 0$. $\therefore\ e^y = x + \sqrt{(x^2+1)}$
Taking *ln* of both sides:	$\Rightarrow y = \ln\left[x + \sqrt{x^2+1}\right] = \operatorname{arsinh}(x)$

Identities and equations

You will need to be able to use and prove hyperbolic identities, which are very similar to their trigonometric counterparts. These can all be proved using the exponential forms of the hyperbolic functions. Here are the most important ones, from which any others can be derived.

- $\sinh(A \pm B) \equiv \sinh(A)\cosh(B) \pm \cosh(A)\sinh(B)$
- $\cosh(A \pm B) \equiv \cosh(A)\cosh(B) \pm \sinh(A)\sinh(B)$
- $\cosh^2 A - \sinh^2 A \equiv 1$
- $\cosh 2A \equiv 2\cosh^2 A - 1 \equiv 1 + 2\sinh^2 A$
- $\sinh 2A = 2\sinh A \cosh A$

Generally, you can use what is known as Osborn's rule to find the hyperbolic identity corresponding to a trigonometric identity. Osborn's rule tells us that given a trigonometric identity, you can replace *sin* by *sinh* and *cos* by *cosh*, but a product of two *sin* terms or, an implied product of two *sin* terms, must be replaced by the negative of the product of two *sinh* terms. For example,

$\Rightarrow$ $\cos 2x = 2\cos^2 x - 1 \rightarrow \cosh 2x = 2\cosh^2 x - 1$ — Replacing *cos* by *coshx*

$\Rightarrow$ $\cos 2x = 1 - 2\sin^2 x \rightarrow \cosh 2x = 1 + 2\sinh^2 x$ — Replacing *sin* with *sinhx*

$\Rightarrow$ $\tan^2 A \rightarrow -\tanh^2 A$ — The LHS Is an implied product of two *sin* terms, because while *sin* isn't explicitly written, we know that $\tan^2 A = \frac{\sin^2 A}{\cos^2 A}$.

Example 2: Prove that $\cosh(2A) \equiv 2\cosh^2 A - 1$.

When proving hyperbolic identities, you should use the exponential definitions of the hyperbolic functions. Start with the *RHS*.	$RHS = 2\cosh^2 A - 1 = 2\left(\frac{e^x + e^{-x}}{2}\right)^2 - 1$ $= \frac{e^{2x} + e^{-2x} + 2}{2} - 1$
Rewrite 1 as $\frac{2}{2}$ and express everything as one fraction.	$= \frac{e^{2x} + e^{-2x} + 2}{2} - \frac{2}{2}$ $= \frac{e^{2x} + e^{-2x}}{2}$
This is equivalent to cosh $2x$, as required.	$= \cosh 2x = LHS$

Example 3: Solve $\cosh 2x - 5\cosh x + 4 = 0$, giving your answers as natural logarithms where possible.

Use $\cosh(2x) \equiv 2\cosh^2 x - 1$	$2\cosh^2 x - 1 - 5\cosh x + 4 = 0$ $2\cosh^2 x - 5\cosh x + 3 = 0$
This is a quadratic in $\cosh x$. Use the quadratic formula with $a = 2,\ b = -5,\ c = 3$:	$\cosh x = \frac{3}{2},\ \cosh x = 1$ So $x = \operatorname{arcosh}\left(\frac{3}{2}\right), x = \operatorname{arcosh}(1)$
Use $\operatorname{arcosh}(x) = \ln\left[x + \sqrt{x^2 - 1}\right]$ with $x = \frac{3}{2}, x = 1$. Note that for $x = \frac{3}{2}$ we will have two solutions because $\frac{3}{2} > 1$.	$x = \ln\left[\frac{3}{2} \pm \sqrt{\frac{9}{4} - 1}\right] = \ln\left[\frac{3}{2} \pm \frac{\sqrt{5}}{2}\right]$ $x = \ln\left[1 + \sqrt{1 - 1}\right] = 0$

Differentiating hyperbolic functions

The following results can be used to differentiate hyperbolic functions:

- $\frac{d}{dx}(\sinh x) = \cosh x$
- $\frac{d}{dx}(\cosh x) = \sinh x$
- $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

You can be expected to use any techniques you learnt from Chapter 9 of Pure Year 2 (Differentiation) to differentiate hyperbolic functions. You also need to be able to prove and use the following results for the inverse hyperbolic functions:

- $\frac{d}{dx}(\operatorname{arsinh} x) = \frac{1}{\sqrt{x^2 + 1}}$
- $\frac{d}{dx}(\operatorname{arcosh} x) = \frac{1}{\sqrt{x^2 - 1}}, \quad x > 1$
- $\frac{d}{dx}(\operatorname{artanh} x) = \frac{1}{1 - x^2}, \quad |x| < 1$

Example 4: Show that $\frac{d}{dx}(\operatorname{arsinh} x) = \frac{1}{\sqrt{x^2+1}}$.

Let $y = \operatorname{arsinh} x$ and take *sinh* of both sides.	$y = \operatorname{arsinh} x$ $\therefore x = \sinh y$
Differentiate both sides with respect to y.	$\frac{dx}{dy} = \cosh y$
But since $\cosh^2 u - \sinh^2 u \equiv 1$ and $x = \sinh y$, we have that $\cosh y = \sqrt{1 + x^2}$.	$\therefore \frac{dx}{dy} = \sqrt{1 + x^2}$
Take the reciprocal of both sides.	$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}$

Integrating hyperbolic functions

Finally, you need to be confident using hyperbolic functions when integrating. The following results are important:

- $\int \sinh x\ dx = \cosh x + c$
- $\int \cosh x\ dx = \sinh x + c$
- $\int \tanh x\ dx = \ln\cosh x + c$
- $\int \frac{1}{\sqrt{x^2 - a^2}}\ dx = \operatorname{arcosh}\left(\frac{x}{a}\right) + c\ ,\ \ x > a$ (I)
- $\int \frac{1}{\sqrt{a^2 + x^2}}\ dx = \operatorname{arsinh}\left(\frac{x}{a}\right) + c$ (II)

You also need to be able to use hyperbolic substitutions to prove the results marked (I) and (II) above, as well as to integrate other expressions that are similar in form. If you are not told what substitution to use, then it is helpful to remember:

- For an integral involving $\sqrt{x^2 + a^2}$, try $x = a\sinh u$.
- For an integral involving $\sqrt{x^2 - a^2}$, try $x = a\cosh u$.

Example 5: Find $\int \sqrt{1 + x^2}\ dx$.

Use the substitution $x = \sinh u$:	$\frac{dx}{du} = \cosh u \therefore dx = \cosh u\ du$
Use $\cosh^2 u - \sinh^2 u \equiv 1$	$\int \sqrt{1 + x^2}\ dx = \int \sqrt{1 + \sinh^2 u}\cosh u\ du$. $\int \sqrt{\cosh^2 u}\cosh u\ du = \int \cosh^2 u\ du$
Since $\cosh 2u = 2\cosh^2 u - 1$, we have that $\cosh^2 u = \frac{\cosh 2u + 1}{2}$. Use $\sinh 2u = 2\sinh u \cosh u$ to simply the result.	$= \frac{1}{2}\int \cosh 2u + 1\ du = \frac{1}{2}\left[\frac{1}{2}\sinh 2u + u\right] + c$ $= \frac{1}{2}\sinh u \cosh u + \frac{1}{2}u + c$
Since $x = \sinh u$, $u = \operatorname{arsinh}(x)$. We use this to write our result in terms of x. Note that $\cosh u \equiv \sqrt{1 + \sinh^2 u} = \sqrt{1 + x^2}$.	$= \frac{1}{2}\operatorname{arsinh}(x) + \frac{1}{2}x\sqrt{1 + x^2} + c$